Introduction to Computational Plasma Physics

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Computational Plasma Physics: Uniquely Challenging

Vast majority of plasma physics in contained in the Vlasov-Maxwell equations that describe self-consistent evolution of distribution function $f(\mathbf{x}, \mathbf{v}, t)$ and electromagnetic fields:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = \left(\frac{\partial f_s}{\partial t} \right)_c$$

where $\mathbf{F}_s = q_s/m_s(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The EM fields are determined from Maxwell equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$
$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \sum_s q_s \int_{-\infty}^{\infty} v f_s \, d\mathbf{v}^3$$

Highly nonlinear: fields tell particles how to move. Particle motion generates fields. This is a very difficult system of equations to solve! Theoretical and computational plasma physics consists of making approximations and solving these equations in specific situations.

Why is solving Vlasov-Maxwell equations hard?

Despite being the fundamental equation in plasma physics the VM equations remain highly challenging to solve.

- Highly nonlinear with the coupling between fields and particles via currents and Lorentz force. Collisions can further complicate things due to long-range forces in a plasma; dominated by small-angle collisions
- High dimensionality and multiple species with large mass ratios: 6D phase-space, $m_e/m_p = 1/1836$ and possibly dozens of species.
- Enormous scales in the system: light speed and electron plasma oscillations; cyclotron motion of electrons and ions; fluid-like evolution on intermediate scales; resistive slow evolution of near-equilibrium states; transport scale evolution in tokamak discharges. 14 orders of magnitude of physics in these equations!

Goals and Outline of Talk

Goal of this talk is to introduce modern concepts in computational plasma physics, specially with a view towards connecting *continous* and *discrete* properties of the equations.

- **Part 1**: Physics that should be preserved in the discrete system. Indirect properties and going beyond accuracy and order of schemes.
- **Part 2:** Overview of a schemes for kinetic equations: Particle-in-Cell (PIC) and continuum solutions to Vlasov equations.

Why Care?

One can look at computational physics in two ways: as an end in itself, and as a tool for applications. Both of these are important!

As end in itself:

- Sits between applied mathematics and theoretical physics. The goal is to design efficient numerical methods to solve equations from theoretical physics.
- The goal here is the numerical method itself: what are its properties? Does it faithfully represent the underlying physics? Does it run efficiently on modern computers? Research into modern numerical methods (including structure preserving methods) fall into this category.
- Usually, besides the fun of solving complex equations (and writing code), the goal is to gain deeper understanding of underlying physics. **Some theoretical questions can only be answered with computer simulations.**

Why Care?

As a tool for applications:

- The second is to look at the computational physics as providing tools to understand/design experiments or observations.
- Large number of routine calculations are needed to build modern experiments (heat-transfer, structural analysis, basic fluid mechanics, equilibrium and stability calculations, etc). **Such routine calculations are no longer cutting edge research topics.**

However, today strong need to be at **intersection of cutting-edge computational physics and critical applications**: E.g: More than $4 billion are invested in private fusion efforts; billions more in public efforts. Low-temperature plasmas also industrially important, and have serious numerical challenges.

Exploring the Fusion Design Space

What is needed to explore or "confine" the design space for the crowded space of fusion concepts?

- Unfortunately, neither the physical models or the numerics are fully developed yet to understand *burning plasma regime*. Enormous scales, hairy plasma-material-interaction and zoo of possible instabilities.
- Two approaches: do biggest possible calculations with first-principles models on the biggest computers. Probably not feasible as a *design* tool!
- Do many calculations in an *optimization loop* to confine the design space. Run occasional large calculations to verify.

I think the latter is the best approach: needs development of appropriate reduced models, fast numerical methods on modern hardware arhitecture. Put everything in an optimization loop.

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The Physics of Discretized Equations

One view of computational physics: we are studying the *physics of discretized equations* and not really "Nature" itself. Not obvious that these are the same (as measure by some metric).

Many important physical properties are "indirect". Simplest example:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} = 0.$$

From this we can derive a conservation law for L_2 norm:

$$\frac{\partial}{\partial t}\frac{1}{2}f^2 + \frac{\partial}{\partial x}\frac{1}{2}f^2 = 0.$$

This is an example of an "indirect" property. **Not obvious** that the L_2 norm of your discrete solution is actually preserved by the scheme you choose. In fact, **not obvious** if it even *should* be preserved!

The Physics of Discretized Maxwell Equations

Maxwell equations have very important *indirect* properties: conservation of momentum and energy. Energy conservation:

$$\frac{\partial}{\partial t}\left(\frac{\mathrm{B}^2}{2\mu_0}+\frac{\epsilon_0\mathrm{E}^2}{2}\right)+\frac{1}{\mu_0}\nabla\cdot(\mathrm{E}\times\mathrm{B})=-\mathrm{E}\cdot\mathrm{J}.$$

- Does a scheme conserve the *discrete* energy? Should it conserve discrete energy?
- Choices based on tradeoff one is willing to make.

Effect of Discrete Energy Conservation

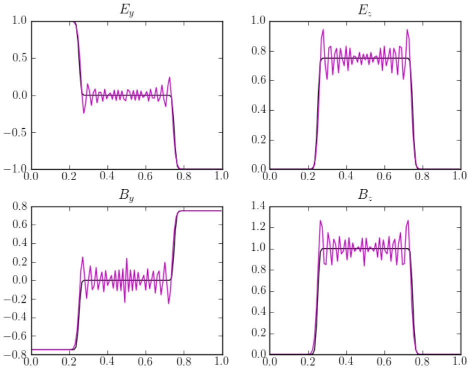

Figure: Energy conserving scheme for Maxwell equations (purple) compared to non-conservative scheme (black). Conservation of energy means there is no damping of spurious high-k modes.

Energy Conservation in Vlasov-Maxwell is Indirect

If we evolve the Vlasov equation we are evolving the distribution function $f(\mathrm{x}, \mathrm{v}, t)$ in phase-space. However, total energy is a *moment* of the distribution function and also includes electromagnetic terms.

$$\frac{d}{dt}\sum_s \int_K \frac{1}{2} m|\mathrm{v}|^2 f \, dz + \frac{d}{dt}\int_\Omega \left(\frac{\epsilon_0}{2}|\mathrm{E}|^2 + \frac{1}{2\mu_0}|\mathrm{B}|^2 \right) \, d^3\mathrm{x} = 0$$

Not obvious that a scheme will conserve total energy. Similar issues in plasmas described by Hamiltonian equations: energy conservation is indirect and depends on the ability of numerical scheme to preserve properties of the *Poisson Bracket* operator.

The Question of Discrete Entropy

In addition to momentum and energy, the Vlasov-Maxwell and other equations have an entropy that is either constant (collisionless plasma) or increases monotonically (collisional plasmas)

$$\frac{d}{dt}\int_K -f\ln(f) \geq 0.$$

Again, it is not immediately clear how the entropy of the discrete system behaves.

A More Subtle Form of Indirect Properties

Even if you *have* an explicit conservation law, there are other subtle properties that are indirect. For example, typical fluid codes will evolve total energy equation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot [\mathrm{u}(\mathcal{E} + p)] = 0.$$

where total energy $\mathcal{E}$ contains two contributions, one from the kinetic energy and the other from the internal energy:

$$\mathcal{E} = \frac{1}{2}\rho \mathrm{u} \cdot \mathrm{u} + \frac{p}{\gamma - 1}$$

The Behaviour of Kinetic Energy

Now consider the evolution of just the *kinetic energy*

$$\frac{\partial}{\partial t}\int_\Omega \frac{1}{2}\rho u^2\, d\mathrm{x} + \oint_{\partial\Omega}\left(\frac{1}{2}\rho u^2 + p\right)\mathrm{u}\cdot\, d\mathrm{s} - \int_\Omega p\nabla\cdot\mathrm{u}\, d\mathrm{x} = 0.$$

- This equation shows that the KE in a volume only changes due to the compressibility of the fluid.
- Similarly, in a plasma, there will be $\mathrm{J}\cdot\mathrm{E}$ term that allows energy exchange between particles and fields.
- Question: does the numerical scheme have this property? That is, is there any suprious exchange terms that are messing up the physics of energy partition between the various terms?

An incorrect discrete exchange can lead to improper behaviour of energy at the highest-k modes. These modes are precisely what we need to get correct to understand turbulence!

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Single particle motion in an electromagnetic field

- In the *Particle-in-cell* (PIC) method the Vlasov-Maxwell equation is solved in the *Lagrangian frame* in which the phase-space is represented by *finite-sized* "macro-particles".
- In the Lagrangian frame the distribution function remains constants along *characteristics* in phase-space.
- These characteristics satisfy the ODE of particles moving under Lorentz force law

$$\frac{d\mathrm{x}}{dt} = \mathrm{v}$$
$$\frac{d\mathrm{v}}{dt} = \frac{q}{m}(\mathrm{E}(\mathrm{x}, t) + \mathrm{v} \times \mathrm{B}(\mathrm{x}, t))$$

- We will first focus on solving the equations-of-motion for the macro-particles, leaving solution of Maxwell equations and coupling to particles for later.

Simple harmonic oscillator

Consider first the simple harmonic oscillator

$$\frac{d^2z}{dt^2} = -\omega^2 z$$

This has exact solution $z = a\cos(\omega t) + b\sin(\omega t)$, where a and b are arbitrary constants. How to solve this numerically? Write as a system of first-order ODEs

$$\frac{dz}{dt} = v; \quad \frac{dv}{dt} = -\omega^2 z$$

Note that the coordinates (z, v) label the *phase-space* of the harmonic oscillator. Multiply the second equation by v and use the first equation to get

$$\frac{d}{dt}\left(\frac{1}{2}v^2 + \frac{1}{2}\omega^2 z^2\right) = 0.$$

This is the *energy* and is *conserved*.
Question: how to solve the ODE such that the energy is conserved by the discrete scheme?

Harmonic oscillator: Forward Euler Scheme

First attempt: use the simplest possible scheme, replace derivatives with difference approximations

$$\frac{z^{n+1}-z^n}{\Delta t} = v^n; \quad \frac{v^{n+1}-v^n}{\Delta t} = -\omega^2 z^n$$

or

$$z^{n+1} = z^n + \Delta t v^n; \quad v^{n+1} = v^n - \Delta t \omega^2 z^n$$

This is the *forward Euler* scheme. Lets check if the discrete scheme conserves energy:

$$(v^{n+1})^2 + \omega^2 (z^{n+1})^2 = (1+\omega^2\Delta t^2)((v^n)^2 + \omega^2 (z^n)^2)$$

The presence of the $\omega^2 \Delta t^2$ in the bracket spoils the conservation. So the forward Euler scheme *does not* conserve energy. Also, note that the energy, in fact, is *increasing*!

Harmonic oscillator: Forward Euler Scheme

Closer look: write as a matrix equation

$$\begin{bmatrix} z^{n+1} \\ v^{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ -\omega^2 \Delta t & 1 \end{bmatrix}}_{\text{Jacobian, } J} \begin{bmatrix} z^n \\ v^n \end{bmatrix}.$$

Observe that the determinant of the Jacobian is $\det(J) = (1 + \omega^2 \Delta t^2)$ which is the same factor as appears in the energy relation. One may reasonably conjecture that when this determinant is one, then perhaps energy is conserved.

Volume Preserving Scheme

We will call say a scheme preserves *phase-space* volume if the determinant of the Jacobian is $\det(J) = 1$.

Harmonic oscillator: Mid-point Scheme

Perhaps a better approximation will be obtained if we use *averaged* values of z, v on the RHS of the discrete equation:

$$\frac{z^{n+1} - z^n}{\Delta t} = \frac{v^n + v^{n+1}}{2}$$
$$\frac{v^{n+1} - v^n}{\Delta t} = -\omega^2 \frac{z^n + z^{n+1}}{2}$$

This is an *implicit* method as the solution at the next time-step depends on the old as well as the next time-step values. In this simple case we can explicitly write the update in a matrix form as

$$\begin{bmatrix} z^{n+1} \\ v^{n+1} \end{bmatrix} = \frac{1}{1+\omega^2\Delta t^2/4} \begin{bmatrix} 1-\omega^2\Delta t^2/4 & \Delta t \\ -\omega^2\Delta t & 1-\omega^2\Delta t^2/4 \end{bmatrix} \begin{bmatrix} z^n \\ v^n \end{bmatrix}.$$

For this scheme $\det(J) = 1$. So the mid-point scheme conserves phase-space volume! Some algebra also shows that

$$(v^{n+1})^2 + \omega^2 (z^{n+1})^2 = (v^n)^2 + \omega^2 (z^n)^2$$

showing that energy is also conserved by the mid-point scheme.

Harmonic oscillator: Mid-point Scheme is symplectic

A more stringent constraint on a scheme for the simple harmonic oscillator is that it be *symplectic*. To check if a scheme is symplectic one checks to see if

$$J^T \sigma J = \sigma$$

where σ is the *unit symplectic matrix*

$$\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Turns out that the mid-point scheme for the harmonic oscillator is also symplectic. Note that if a scheme conserves phase-space volume, it *need not* be symplectic.

Accuracy and Stability

To study the stability, accuracy and convergence of a scheme one usually looks at the first order ODE

$$\frac{dz}{dt} = -\gamma z$$

where $\gamma = \lambda + i\omega$ is the complex frequency. The exact solution to this equation is $z(t) = z_0 e^{-\gamma t}$. The solution has damped/growing modes ($\lambda > 0$ or $\lambda < 0$) as well as oscillating modes.

- The forward Euler scheme for this equation is

$$z^{n+1} = z^n - \Delta t \gamma z^n = (1 - \Delta t \gamma) z^n.$$

- The mid-point scheme for this equation is

$$z^{n+1} = \left(\frac{1 - \gamma \Delta t/2}{1 + \gamma \Delta t/2} \right) z^n$$

Accuracy and Stability

We can determine how *accurate* the scheme is by looking at how many terms the scheme matches the Taylor series expansion of the exact solution:

$$z(t^{n+1}) = z(t^n)\left(1 - \gamma\Delta t + \frac{1}{2}\gamma^2\Delta t^2 - \frac{1}{6}\gamma^3\Delta t^3 + \dots\right)$$

- The forward Euler scheme matches the *first two terms*

$$z^{n+1} = z^n(1 - \Delta t\gamma)$$

- The mid-point scheme matches the *first three terms*

$$z^{n+1} = z^n\left(1 - \Delta t\gamma - \frac{1}{2}\gamma^2\Delta t^2 - \frac{1}{4}\gamma^3\Delta t^3 + \dots\right)$$

Accuracy and Stability

We can determine if the scheme is *stable* by looking at the amplification factor $|z^{n+1}/z^n|$. Note that for damped modes $(\lambda > 0)$ this quantity *decays* in time, while for purely oscillating modes $(\lambda = 0)$ this quantity remains *constant*.

- The amplification factor for the forward Euler scheme in the absence of damping is $1 + \omega^2 \Delta t^2 > 1$, hence this scheme is *unconditionally unstable*.
- The amplification factor for the mid-point scheme in the absence of damping is exactly 1, showing that the mid-point scheme is *unconditionally stable*, that is, one can take as large time-step one wants without the scheme "blowing up". Of course, the errors will increase with larger Δt.

Runge-Kutta schemes

- Even though the forward Euler scheme is unconditionally unstable, we can use it to construct other schemes that *are* stable and are also more accurate (than first order).
- For example, a class of Runge-Kutta schemes can be written as a combination of forward Euler updates. In particular, the *strong stability preserving* schemes are important when solving hyperbolic equations. Note that these RK schemes will *not* conserve energy for the harmonic oscillator, but *decay* it.
- Other multi-stage Runge-Kutta schemes can be constructed that allow very large time-steps for diffusive processes, for example, that come about when time-stepping diffusion equations.

Single particle motion in an electromagnetic field

- In PIC method the Vlasov-Maxwell equation is solved in the *Lagrangian frame*: the phase-space is represented by *finite-sized* "macro-particles".
- In the Lagrangian frame the distribution function remains constants along *characteristics* in phase-space.
- These characteristics satisfy the ODE of particles moving under Lorentz force law

$$
\begin{aligned}
\frac{d\mathrm{x}}{dt} &= \mathrm{v} \\
\frac{d\mathrm{v}}{dt} &= \frac{q}{m}(\mathrm{E}(\mathrm{x}, t) + \mathrm{v} \times \mathrm{B}(\mathrm{x}, t))
\end{aligned}
$$

- In the absence of an electric field, the kinetic energy must be conserved

$$\frac{1}{2}|\mathrm{v}|^2 = \mathrm{constant}.$$

This is independent of the spatial or time dependence of the magnetic field. Geometrically this means that in the absence of an electric field the velocity vector rotates and its tip always lies on a sphere.

Single particle motion in an electromagnetic field

- A mid-point scheme for this equation system would look like

$$\frac{\mathrm{x}^{n+1}-\mathrm{x}^n}{\Delta t} = \frac{\mathrm{v}^{n+1}+\mathrm{v}^n}{2}$$

$$\frac{\mathrm{v}^{n+1}-\mathrm{v}^n}{\Delta t} = \frac{q}{m}\left(\overline{\mathrm{E}}(\mathrm{x},t) + \frac{\mathrm{v}^{n+1}+\mathrm{v}^n}{2}\times\overline{\mathrm{B}}(\mathrm{x},t)\right)$$

  The overbars indicate some averaged electric and magnetic fields evaluated from the new and old positions. In general, this would make the scheme nonlinear!

- Instead, we will use a *staggered* scheme in which the position and velocity are staggered by half a time-step.

$$\frac{\mathrm{x}^{n+1}-\mathrm{x}^n}{\Delta t} = \mathrm{v}^{n+1/2}$$

$$\frac{\mathrm{v}^{n+1/2}-\mathrm{v}^{n-1/2}}{\Delta t} = \frac{q}{m}\left(\mathrm{E}(\mathrm{x}^n,t^n) + \frac{\mathrm{v}^{n+1/2}+\mathrm{v}^{n-1/2}}{2}\times\mathrm{B}(\mathrm{x}^n,t^n)\right)$$

The Boris algorithm for the staggered scheme

The velocity update formula is

$$\frac{\mathbf{v}^{n+1/2}-\mathbf{v}^{n-1/2}}{\Delta t}=\frac{q}{m}\left(\mathbf{E}(\mathbf{x}^n,t^n)+\frac{\mathbf{v}^{n+1/2}+\mathbf{v}^{n-1/2}}{2}\times\mathbf{B}(\mathbf{x}^n,t^n)\right)$$

This appears like an implicit method: most obvious is to construct a linear 3×3 system of equations and invert them to determine $\mathbf{v}^{n+1}$. Puzzle to test your vector-identity foo: find $\mathbf{A}$ if $\mathbf{A}=\mathbf{R}+\mathbf{A}\times\mathbf{B}$.

The Boris algorithm updates this equation in three steps:

$$\begin{aligned}
\mathbf{v}^- &= \mathbf{v}^{n-1/2}+\frac{q}{m}\mathbf{E}^n\frac{\Delta t}{2}\\
\frac{\mathbf{v}^+-\mathbf{v}^-}{\Delta t} &= \frac{q}{2m}(\mathbf{v}^++\mathbf{v}^-)\times\mathbf{B}^n\\
\mathbf{v}^{n+1/2} &= \mathbf{v}^++\frac{q}{m}\mathbf{E}^n\frac{\Delta t}{2}
\end{aligned}$$

Convince yourself that this is indeed equivalent to the staggered expression above. So we have two electric field updates with half time-steps and a rotation due to the magnetic field. Once the updated velocity is computed, we can trivially compute the updated positions.

The Boris algorithm for the staggered scheme

How to do the rotation? The Boris algorithm does this in several steps:

- Compute the t and s vectors as follows

$$\mathrm{t} = \frac{q\mathrm{B}}{m}\frac{\Delta t}{2}$$

$$\mathrm{s} = \frac{2\mathrm{t}}{1+|\mathrm{t}|^2}$$

- Compute $\mathrm{v}' = \mathrm{v}^- + \mathrm{v}^- \times \mathrm{t}$ and finally $\mathrm{v}^+ = \mathrm{v}^- + \mathrm{v}' \times \mathrm{s}$.

See Birdsall and Langdon text book Section 4-3 and 4-4 and figure 4-4a. Easily extended to relativistic case.

Note that in the absence of an electric field the Boris algorithm conserves kinetic energy.

Why is Boris algorithm good? Can one do better?

See paper by Qin at. al. Phys. Plasmas, **20**, 084503 (2013) in which it is shown that the Boris algorithm *conserves phase-space volume*. However, they also show that the Boris algorithm is *not* symplectic.

- The relativistic Boris algorithm does not properly compute the $\mathrm{E} \times \mathrm{B}$ velocity. This can be corrected. For example Vay, Phys. Plasmas, **15**, 056701 (2008). The Vay algorithm however, breaks the phase-space volume preserving property of the Boris algorithm.
- Higuera and Cary, Phys. Plasmas, **24**, 052104 (2017) showed how to compute the correct $\mathrm{E} \times \mathrm{B}$ drift velocity and restore volume preserving property. Seems this is probably the current-best algorithm for updating Lorentz equations.
- The saga for better particle push algorithms is not over! For example, an active area of research is to discover good algorithms for *asymptotic* systems, for example, when gyroradius is much smaller than gradient length-scales or gyrofrequency is much higher than other time-scales in the system. Common in most magnetized plasmas.

Solving Maxwell equations

Besides pushing particles in electromagnetic fields, we need to compute these fields self-consistently from currents and charges by solving Maxwell equations. First consider Maxwell equations in vacuum:

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$
$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = 0$$

For these we have the conservation laws

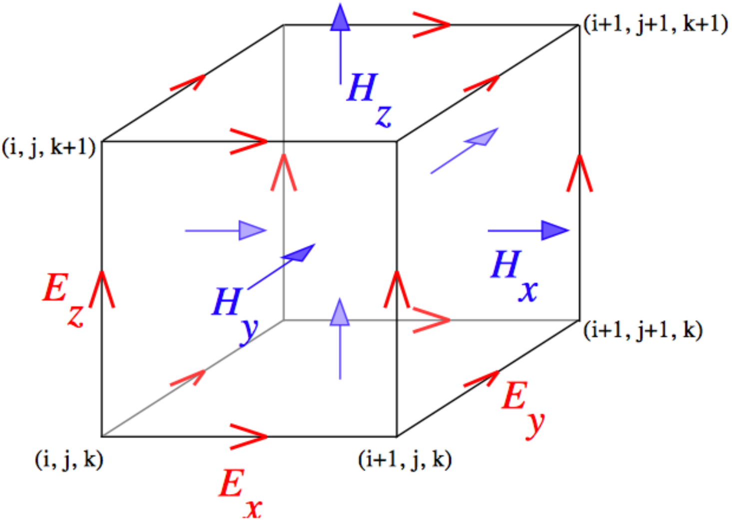
$$\frac{d}{dt} \int_\Omega \mathbf{E} \times \mathbf{B} \, d^3\mathbf{x} = 0$$
$$\frac{d}{dt} \int_\Omega \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) d^3\mathbf{x} = 0.$$

Note these are *global* conservation laws and one can instead also write *local* conservation laws that include momentum and energy flux terms. How to solve these equations efficiently and maintain (some) conservation and geometric properties?

Solving Maxwell equations

- Maxwell equations have a very special geometric structure. The electric field E is a *vector* while the magnetic field B is a *bivector* (this is disguised in the usual formulations of Maxwell equations).
- (In spacetime formulations the complete electromagnetic field is represented as a single bivector in 4D space-time).
- The fact that we are dealing with two objects of *different* geometric types indicates that the discrete Maxwell equations should also inherit this somehow.
- The Yee algorithm, often called the *finite-difference time-domain* algorithm is the most successful (and simple) algorithm that accounts of this geometric structure. It is implemented in most PIC codes, though recent research has focused on structure preserving finite-element and other methods.

Solving Maxwell equations: The Yee-cell



Solving Maxwell equations: The Yee-cell

On the Yee-cell the difference approximation to Maxwell equations “falls out”, almost like magic. The updates are staggered in time and use two *different* discrete curl operators:

$$\mathrm{B}^{n+1/2} = \mathrm{B}^{n-1/2} - \Delta t\, \nabla_E \times \mathrm{E}^n$$
$$\mathrm{E}^{n+1} = \mathrm{E}^n + \Delta t/c^2\, \nabla_F \times \mathrm{B}^{n+1/2}$$

Here the symbols $\nabla_F \times$ and $\nabla_E \times$ are the discrete curl operators:

- The first takes *face-centered* magnetic field and computes it curl. This operator *puts the result on cell edges*.
- The second takes *edge-centered* electric field and computes it curl. This operator *puts the result on cell faces*.
- The structure of Yell-cell also indicates that *currents* must be co-located with the electric field and computed at half time-steps.

This duality neatly reflects the underlying geometry of Maxwell equations. The staggering in time reflects the fact that in 4D the electromagnetic field is a bivector in spacetime.

Divergence relations are exactly maintained

We can show that the discrete Maxwell equations on a Yee-cell maintain the divergence relations exactly:

$$\nabla_F \cdot \mathrm{B}^{n+1/2} = 0$$
$$\nabla_E \cdot \mathrm{E}^{n} = 0.$$

There is an additional constraint of Maxwell equations in a plasma, that is, the current conservation:

$$\frac{\partial \varrho_c}{\partial t} + \nabla \cdot \mathrm{J} = 0.$$

where ϱ_c is the charge density and J is the current density. On the Yell-cell this becomes

$$\frac{\varrho_c^{n+1} - \varrho_v^{n}}{\Delta t} + \nabla_E \cdot \mathrm{J}^{n+1/2} = 0.$$

One must ensure that current from particles is computed carefully to ensure that this expression is satified. See Esirkepov, Comp. Phys. Communications, **135** 144-153 (2001).

Conclusion and On-Going Work

- Computational plasma physics is hard: much work is remaining to create tools for the variety of plasma applications; subtle indirect properties must be accounted for in design of schemes.
- Potentially high-impact field: need to confine space of possible fusion devices and help understand anticipated burning-plasma regime.

Some Parting Thoughts and All the Best!

Here are some personal parting thoughts on computational physics:

- Computational physics is a rapidly evolving field. Good field to be in!
- Strive for **technical excellence**. Do not settle for existing methods or tools and spend time in understanding deeply **both** the physics of the equations *and* the numerics used to solve them. Go beyond your classwork and thesis reasearch (make it a point to read arxiv physics.comp-ph and math.NA postings *every day*).
- Modern computational physics is moving to C and C++: please learn them. Use good software practices (write modular code, use version control, build systems, regression tests). Even for your thesis code!
- To become *really good* you must **apprentice yourself** to a genuine expert.
- If you become an expert at the (i) physics (ii) mathematics of the numerical methods (iii) programming and software techniques, you will be in a very strong position to contribute to development in many different fields. You will bring *unique skills* which few other people will be