Lecture IV: A Tutorial Introduction to Discontinuous Galerkin Schemes

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What are discontinuous Galerkin schemes?

Discontinuous Galerkin schemes are a class of *Galerkin* schemes in which the solution is represented using *piecewise discontinuous* functions.

- *Galerkin* minimization
- Piecewise *discontinuous* representation

Goal of this lecture is to understand conceptual meaning of discontinuous Galerkin schemes and understand how to use them to solve PDEs. Much is left out as the literature on DG is vast, but will aim to cover key conceptual ideas. Outline

- Discontinuous Galerkin representation, recovery and weak-equalities
- DG scheme for linear advection and extension to Maxwell equations. Aspects of DG for nonlinear problems
- Application of DG to plasma kinetic equations

Discontinuous Galerkin algorithms represent state-of-art for solution of hyperbolic partial differential equations

DG algorithms hot topic in CFD and applied mathematics.

- First introduced by Reed and Hill in 1973 as a conference paper to solve steady-state neutron transport equations. More than 2100 citations.
- Some earlier work on solving *elliptic* equations by Nitsche in 1971 (original paper in German). Introduced the idea of "interior penalty". Usually, though, DG is not used for elliptic problems. Paradoxically, perhaps DG may be even better for certain elliptic/parabolic problems.
- Key paper for nonlinear systems in multiple dimensions is by Cockburn and Shu (JCP, **141**, 199-224, 1998). More than 1700 citations.
- Almost continuous stream of papers in DG, both for fundamental formulations and applications to physics and engineering problems. This continues to be an active area of research, and at present *DG is under-utilized in plasma physics*.

Essential idea of Galerkin methods: L_2 minimization of errors on a finite-dimensional subspace

There two key ingredients in a Galerkin scheme: selection of a finite-dimensional space of functions and a definition of errors.

- Consider a interval $[-1, 1]$. On this, we can choose Legendre polynomials $P_l(x)$ up to some order $l < N$ as a *basis-set*.
- We need to *define* a way to measure errors on this function space. One way to do this is to select an inner product and then use it to define a norm. For example consider the inner-product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\, dx$$

using which we can define the L_2 norm

$$\|f\|_2 = (f, f)$$

Once we have selected the finite-dimensional space of functions and a norm, we can use it to construct a Galerkin method.

Essential idea of Galerkin methods: L_2 minimization of errors on a finite-dimensional subspace

Consider a general time-dependent problem on $x \in [-1, 1]$:

$$f'(x,t) = G[f]$$

where $G[f]$ is some operator. To approximate it expand $f(x)$ with our basis functions $P_k(x)$,

$$f(x,t) \approx f_h(x,t) = \sum_{k=1}^{N} f_k(t) P_k(x)$$

This gives discrete system

$$\sum_{k=1}^{N} f_k' P_k(x) = G[f_h]$$

Question

How to determine f_k' in an optimum manner?

Essential idea of Galerkin methods: L_2 minimization of errors on a finite-dimensional subspace

Answer: Do an L_2 minimization of the error, i.e. find f'_k such that the error as *defined by our selected norm* is minimized.

$$E_N = \left\| \sum_{k=1}^{N} f'_k P_k(x) - G[f_h] \right\|_2 = \int_{-1}^{1} \left[\sum_{k=1}^{N} f'_k P_k(x) - G[f_h] \right]^2 dx$$

For minimum error $\partial E_N / \partial f'_m = 0$ for all $k = 1, \ldots, N$. This leads to the linear system that determines the coefficients f'_k

$$\int_{-1}^{1} P_m(x) \left(\sum_{k=1}^{N} f'_k P_k(x) - G[f_h] \right) dx = 0$$

for all $m = 1, \ldots, N$. This will give

$$f'_k = \frac{2k+1}{2} \int_{-1}^{1} P_k(x) G[f_h] \, dx$$

What does a typical L_2 fit look like for Galerkin scheme?

Consider finding the best-fit on finite-dimensional space for the function $f(x) = 3 + (x-0.5)^4 + 2x^3 - 5x^2$ on $x \in [-1,1]$. Choose *normalized* Legendre polynomials as basis functions.

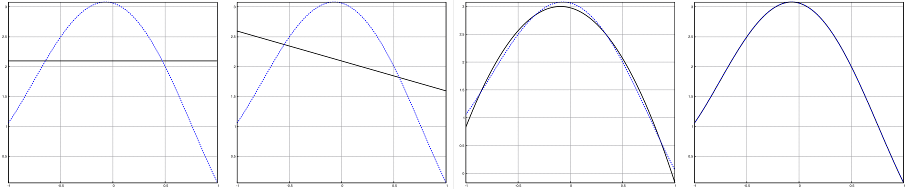


Figure: Best L_2 fit with $p=0$, $p=1$, $p=2$ and $p=4$ for $f(x) = 3 + (x-0.5)^4 + 2x^3 - 5x^2$ on $x \in [-1,1]$.

What does a typical L_2 fit look like for *discontinuous* Galerkin scheme?

In discontinuous Galerkin schemes we split interval into cells and use Galerkin scheme in each cell. This will naturally lead to *discontinuities* across cell boundaries.

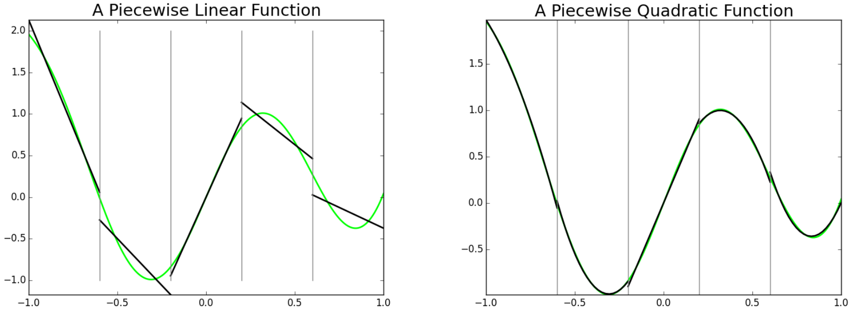


Figure: The best L_2 fit of $x^4 + \sin(5x)$ with piecewise linear (left) and quadratic (right) basis functions.

Weak-equality and recovery

- It is important to remember that the discontinuous Galerkin solution is a *representation* of the solution and not the solution itself.
- Notice that even a continuous function will, in general, have a discontinuous *representation* in DG.

We can formalize this idea using the concept of *weak-equality*. Consider an interval I and select a finite-dimensional function space on it, spanned by basis functions ψ_k, $k = 1, \ldots, N$. Choose an inner product, for example

$$(f, g) \equiv \int_I f(x)g(x)\,dx.$$

Definition (Weak equality)

Two functions, f and g are said to be *weakly equal* if

$$(\psi_k, f - g) = 0$$

for all $k = 1, \ldots, N$. We denote weak equality by

$$f \doteq g.$$

Weak-equality and recovery

- Notice that weak-equality depends on the function space as well as the inner-product we selected.
- The Galerkin L_2 minimization is equivalent to, for example, restating that

  $$f'(x,t) \doteq G[f]$$

  This implies

  $$\big(\psi_k, f'(x,t) - G[f]\big) = 0$$

  which is exactly what we obtained by minimizing the error defined using the L_2 norm.
- Hence, we can say that the *DG scheme only determines the solution in the weak-sense*, that is, all functions that are weakly equal to DG representation can be potentially interpreted as the actual solution.
- This allows a powerful way to construct schemes with desirable properties by *recovering* weakly-equal functions using the DG representations.

Example of recovery: Exponential recovery in a cell

- Consider we have a linear representation of the particle distribution function $f_h(x) = f_0 + x f_1$ in a cell.
- We can use this to *reconstruct* an exponential function that has the desirable property that it is *positive* everywhere in the cell. That is, we want to find

$$\exp(g_0 + g_1 x) \doteq f_0 + x f_1$$

- This will lead to a coupled set of nonlinear equations to determine g_0 and g_1
- Note that this process is not always possible: we need $f_0 > 0$ as well as the condition $|f_1| \leq 3 f_0$. Otherwise, the f_h is not realizable (i.e. there is no positive distribution function with the same moments as f_h).

Example of recovery: Exponential recovery in a cell

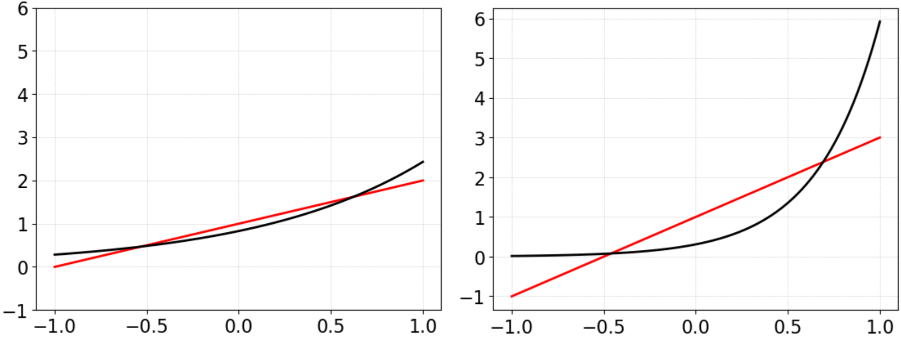


Figure: Recovery of exponential function (black) from linear function (red). Left plot is for $f_0 = 1$, $f_1 = 1$ and right for $f_0 = 1$ and $f_1 = 2$.

Discontinuous Galerkin scheme for linear advection

Consider the 1D passive advection equation on $I \in [L, R]$

$$\frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} = 0$$

with λ the constant advection speed. $f(x,t) = f_0(x - \lambda t)$ is the exact solution, where $f_0(x)$ is the initial condition. Designing a good scheme is much harder than it looks.

- Discretize the domain into elements $I_j \in [x_{j-1/2}, x_{j+1/2}]$
- Pick a finite-dimensional function space to represent the solution. For DG we usually pick polynomials in each cell but allow discontinuities across cell boundaries
- Expand $f(x,t) \approx f_h(x,t) = \sum_k f_k(t) w_k(x)$.

Discrete problem can be stated as finding the coefficients that minimize the L_2 norm of the residual

The discrete problem in DG is stated as: find f_h in the function space such that for each basis function φ we have

$$\int_{I_j} \varphi \left(\frac{\partial f_h}{\partial t} + \lambda \frac{\partial f_h}{\partial x} \right) \, dx = 0.$$

Integrating by parts leads to the discrete *weak-form*

$$\int_{I_j} \varphi \frac{\partial f_h}{\partial t} \, dx + \lambda \varphi_{j+1/2} \hat{F}_{j+1/2} - \lambda \varphi_{j-1/2} \hat{F}_{j-1/2} - \int_{I_j} \frac{d\varphi}{dx} \lambda f_h \, dx = 0.$$

Here $\hat{F} = \hat{F}(f_h^+, f_h^-)$ is the consistent *numerical flux* on the cell boundary. Integrals are performed using high-order quadrature schemes.

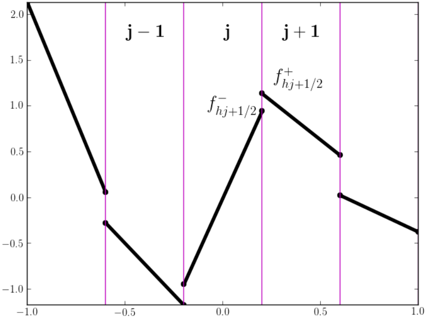
To account for flow across cell-boundary, need to select numerical flux

- Take averages (central fluxes)

$$\hat{F}(f_h^+, f_h^-) = \frac{1}{2}(f_h^+ + f_h^-)$$

- Use upwinding (upwind fluxes)

$$\begin{aligned}\hat{F}(f_h^+, f_h^-) &= f_h^- \quad \lambda > 0 \\ &= f_h^+ \quad \lambda < 0\end{aligned}$$



Example: Piecewise constant basis functions lead to familiar difference equations

- A central flux with piecewise constant basis functions leads to the familiar central difference scheme

$$\frac{\partial f_j}{\partial t} + \lambda \frac{f_{j+1} - f_{j-1}}{2\Delta x} = 0$$

- An upwind flux with piecewise constant basis functions leads to the familiar upwind difference scheme (for $\lambda > 0$)

$$\frac{\partial f_j}{\partial t} + \lambda \frac{f_j - f_{j-1}}{\Delta x} = 0$$

Solution is advanced in time using a suitable ODE solver, usually strong-stability preserving Runge-Kutta methods. (See G2 website)

Example: Piecewise constant basis functions with central flux leads to dispersive errors

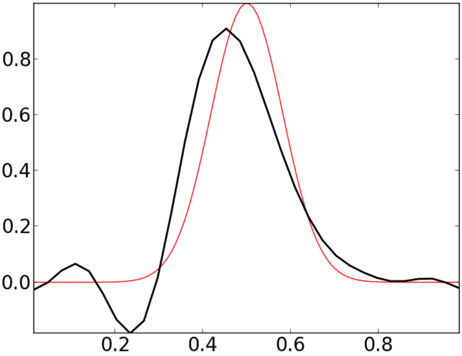


Figure: Advection equation solution (black) compared to exact solution (red) with central fluxes and piecewise constant basis functions.

Example: Piecewise constant basis functions with upwind flux is very diffusive

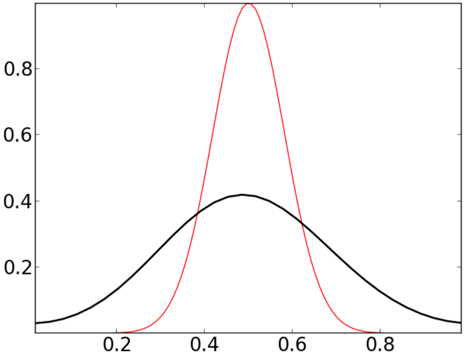


Figure: Advection equation solution (black) compared to exact solution (red) with upwind fluxes and piecewise constant basis functions.

Passive advection with piecewise linear basis functions

To get better results, we can use piecewise linear polynomials instead. That is, select the basis functions

$$\varphi \in \{1, 2(x - x_j)/\Delta x\}$$

In terms of which the solution in each cell is expanded as $f_j(x,t) = f_{j,0} + 2 f_{j,1}(x - x_j)/\Delta x$. With this, some algebra shows that we have the update formulas for *each stage* of a Runge-Kutta method

$$\begin{aligned} f_{j,0}^{n+1} &= f_{j,0}^{n} - \sigma\left(\hat{F}_{j+1/2} - \hat{F}_{j-1/2}\right) \\ f_{j,1}^{n+1} &= f_{j,1}^{n} - 3\sigma\left(\hat{F}_{j+1/2} + \hat{F}_{j-1/2}\right) + 6\sigma f_{j,0} \end{aligned}$$

where $\sigma \equiv \lambda \Delta t/\Delta x$. As these are explicit schemes we need to ensure time-step is sufficiently small. Usually, we need to ensure $\sigma = \lambda \Delta t/\Delta x \le 1/(2p+1)$.

Passive advection with piecewise linear basis functions

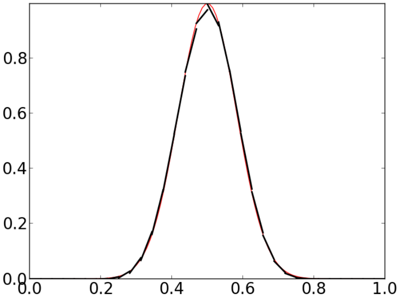


Figure: Advection equation solution (black) compared to exact solution (red) with upwind fluxes and piecewise linear basis functions.

In general, with upwind fluxes and linear basis functions numerical diffusion goes like $|\lambda|\Delta x^3 \partial^4 f / \partial x^4$.

Good numerical methods should inhert some properties from the continuous equations

From the continuous passive advection equation we can show that, on a periodic domain the total particles are conserved

$$\frac{d}{dt}\int_I f\,dx = 0$$

Also, the L_2 norm of the solution is also conserved

$$\frac{d}{dt}\int_I \frac{1}{2}f^2\,dx = 0$$

We would like to know if our discrete scheme *inherits or mimics these properties*. Sometimes, methods in which the discrete scheme inhert important properties from the continuous equations are called *mimetic* methods. However, note that in general it is impossible to inhert *all* properties and often it is not desirable to do so.

To prove properties of the discrete scheme start from discrete weak-form

To understand properties of the scheme we must (obviously) use the *discrete weak-form* as the starting point.

$$\int_{I_j} \varphi \frac{\partial f_h}{\partial t}\, dx + \lambda \varphi_{j+1/2} \hat{F}_{j+1/2} - \lambda \varphi_{j-1/2} \hat{F}_{j-1/2} - \int_{I_j} \frac{d\varphi}{dx} \lambda f_h \, dx = 0.$$

A general technique is to use a function belonging to the *finite-dimensional function space* as the test function φ in the discrete weak-form.
Example: consider we set $\varphi = 1$. Then we get

$$\sum_j \int_{I_j} \frac{\partial f_h}{\partial t}\, dx + \lambda \sum_j \left(\hat{F}_{j+1/2} - \hat{F}_{j-1/2} \right) = 0.$$

The second term sums to zero and so we have shown that

$$\frac{d}{dt} \sum_j \int_{I_j} f_h \, dx = 0.$$

To prove properties of the discrete scheme start from discrete weak-form

Now, consider we use the *solution itself* as the test function. We can do this as the solution, by definition, belongs to the finite-dimensional function space. We get

$$\sum_j \int_{I_j} f_h \frac{\partial f_h}{\partial t}\, dx + \sum_j \left(f^-_{hj+1/2} \hat{F}_{j+1/2} - f^+_{hj-1/2} \hat{F}_{j-1/2} \right) - \sum_j \int_{I_j} \frac{d f_h}{dx} f_h \, dx = 0$$

We can write the last term as

$$\sum_j \int_{I_j} \frac{1}{2} \frac{d}{dx} f_h^2 \, dx = \frac{1}{2} \sum_j \left[\left(f^-_{hj+1/2} \right)^2 - \left(f^+_{hj-1/2} \right)^2 \right]$$

If we use *upwind fluxes* we can show that we get

$$\frac{d}{dt} \sum_j \int_{I_j} f_h^2 \, dx = - \sum_j \left(f^-_{hj+1/2} - f^+_{hj-1/2} \right)^2 \leq 0.$$

Hence, the L_2 norm of the solution *will decay and not remain constant*. However, this is the desirable behavior as it ensures L_2 stability of the discrete system. With central fluxes the L_2 norm is conserved. (Prove this)

Summary of DG schemes for passive advection equation

- Pick basis functions. These are usually piecewise polynomials, but could be other suitable functions.
- Construct discrete weak-form using integration by parts.
- Pick suitable numerical fluxes for the surface integrals.
- Use Runge-Kutta (or other suitable) schemes for evolving the equations in time.
- To prove properties of the scheme, start from the discrete weak-form and use appropriate test-functions and simplify.

For 1D linear hyperbolic systems extension of scheme is straightforward

Consider the 1D source-free Maxwell equations

$$\frac{\partial}{\partial t}\begin{bmatrix} E_y \\ B_z \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} c^2 B_z \\ E_y \end{bmatrix} = 0.$$

Basic idea is to transform the equation into uncoupled advection equations for the *Riemann variables*. This is always possible for *linear hyperbolic* systems. For the above system, multiply the second equation by c and add and subtract from the first equation to get

$$\begin{aligned} \frac{\partial}{\partial t}\left(E_y + cB_z\right) + c\frac{\partial}{\partial x}\left(E_y + cB_z\right) &= 0 \\ \frac{\partial}{\partial t}\left(E_y - cB_z\right) - c\frac{\partial}{\partial x}\left(E_y - cB_z\right) &= 0. \end{aligned}$$

Note that these are two uncoupled passive advection equations for the variables $w^{\pm} = E_y \pm cB_z$ with advection speeds $\pm c$. Hence, we can use scheme for passive advection to construct a scheme for this system.

Choice of numerical fluxes for Maxwell equations impacts energy conservation

The electromagnetic energy is given by

$$\mathcal{E} = \frac{\epsilon_0}{2}E_y^2 + \frac{1}{2\mu_0}B_z^2$$

Notice that this is the L_2 norm of the electromagnetic field.

- Hence, as we showed for the passive advection equation, if we use upwinding to compute numerical fluxes, the *electromagnetic energy will decay*.
- If we use central fluxes (average left/right values) then the EM energy will remain conserved by the *time-continuous* scheme. However, the Runge-Kutta time-stepping will add small diffusion that will decay the total energy a little.
- However, the energy decay rate will be *independent* of the spatial resolution and will reduce with smaller time-steps.

(Run code. Show 2D advection movies before moving to RDG.)

How to discretize parabolic equations with DG?

- DG is traditionally used to solve hyperbolic PDEs. However, DG is also very good for the solution of parabolic PDEs.
- One challenge here is that parabolic PDEs have *second* derivatives and it is not clear at first how a discontinuous representation can allow solving such systems.

Consider the diffusion equation (subscripts represent derivatives)

$$f_t = f_{xx}$$

Choose function space and multiply by test function in this space to get weak form

$$\int_{I_j} \varphi f_t \, dx = \varphi f_x \Big|_{x_{j-1/2}}^{x_{j+1/2}} - \int_{I_j} \varphi_x f_x \, dx.$$

In DG, as f is discontinuous, it is not clear how to compute the derivative across the discontinuity at the cell interface in the first term. (See SimJ JE16).

Lets revisit weak-equality and recovery

Definition (Weak equality)

Two functions, f and g are said to be *weakly equal* if

$$(\psi_k, f - g) = 0$$

for all $k = 1, \ldots, N$. We denote weak equality by

$$f \doteq g.$$

Recall that the DG solution is only a *representation* of the solution and not the solution itself. Hence, we can consider the following "inverse" problem: given a discontinuous solution across two cells, is it possible to *recover* a *continuous representation* that can then be used in the above weak-form?

Use weak-equality to *recover* continuous function

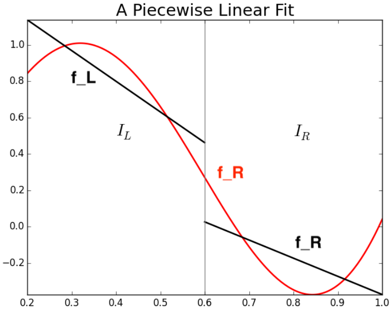


Figure: Given piecewise linear representation (black) we want to recover the continuous function (red) such that **moments of recovered and linear representation are the same in the respective cells**.

Use weak-equality to *recover* continuous function

- Consider recovering $\hat{f}$ on the interval $I = [-1, 1]$, from a function, f, which has a single discontinuity at $x = 0$.
- Choose some function spaces $\mathcal{P}_L$ and $\mathcal{P}_R$ on the interval $I_L = [-1, 0]$ and $I_R = [0, 1]$ respectively.
- Reconstruct a continuous function $\hat{f}$ such that

$$\begin{aligned} \hat{f} &\doteq f_L \quad x \in I_L \quad \text{on } \mathcal{P}_L \\ \hat{f} &\doteq f_R \quad x \in I_R \quad \text{on } \mathcal{P}_R. \end{aligned}$$

  where $f = f_L$ for $x \in I_L$ and $f = f_R$ for $x \in I_R$.

- To determine $\hat{f}$, use the fact that given $2N$ pieces of information, where N is the number of basis functions in $\mathcal{P}_{L,R}$, we can construct a polynomial of maximum order $2N - 1$. We can hence write

$$\hat{f}(x) = \sum_{m=0}^{2N-1} \hat{f}_m x^m.$$

  Plugging this into the weak-equality relations gives a *linear* system for $\hat{f}_m$.

Use recovered function in weak-form

Once we have determined $\hat{f}$ we can use this in the discrete weak-form of the diffusion equation:

$$\int_{I_j} \varphi f_t \, dx = \varphi \hat{f}_x \Big|_{x_{j-1/2}}^{x_{j+1/2}} - \int_{I_j} \varphi_x f_x \, dx.$$

Note that now as $\hat{f}$ is continuous at the cell interface there is no issue in computing its derivative. We can, in fact, do a second integration by parts to get another discrete weak-form

$$\int_{I_j} \varphi f_t \, dx = (\varphi \hat{f}_x - \varphi_x \hat{f}) \Big|_{x_{j-1/2}}^{x_{j+1/2}} + \int_{I_j} \varphi_{xx} f \, dx.$$

This weak-form has certain advantages as the second term does not contain derivatives (which may be discontinuous at cell boundary). (Explicit formula for $p = 0$ case)

Another look at computing numerical fluxes

- To design a scheme for the diffusion equation we used a recovery procedure to compute the edge values and slopes
- Can this process be used to compute numerical fluxes for use in updating advection equations? Potentially much more accurate scheme for smooth solution.

Recall discrete weak-form of advection equation

$$\int_{I_j} \varphi \frac{\partial f_h}{\partial t}\, dx + \lambda \varphi_{j+1/2} \hat{F}_{j+1/2} - \lambda \varphi_{j-1/2} \hat{F}_{j-1/2} - \int_{I_j} \frac{d\varphi}{dx} \lambda f_h\, dx = 0.$$

Instead of using upwinding or central fluxes, we can use recovered polynomial at each cell interface to compute $\hat{F}_{j\pm 1/2}$.

A different way to design a recovery scheme for advection

Consider the general problem of recovering a higher-order function given three neigbor values. We will use the following procedure. Label the three cells I_L, I_0 and I_R.

- First, compute two recovery polynomials. One across the pair of cells I_L, I_0 and another across the cells I_0, I_R.
- Now, construct a polynomial $\hat{f}$ in cell I_0 such that

  $$\hat{f} \doteq f \quad \text{in cell } I_0$$

  and such that the *slopes* of $\hat{f}$ on the edges of the cell match the slopes (and values) of the two recovered polynomials.
- For a p order scheme, we have $p+1+2$ pieces of information and so can recover a $p+2$ order polynomial.

(Run code)

This leads to a *differential* form of the DG scheme

Now that we have the recovered polynomial $\hat{f}$ we can use this *directly* to design a update formula for the advection equation *without* need to derive the discrete weak-form.

$$(\psi_k, f_t) = -(\psi_k, \hat{f}_x)$$

Note that this is system of ODEs for the expansion coefficients f_k and we can update them using Runge-Kutta or other time-steppers. (Run code)

Putting everything together: how to solve the Vlasov-Maxwell equation using DG?

We would like to solve the Vlasov-Maxwell system, treating it as a partial-differential equation (PDE) in 6D:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = 0$$

where $\mathbf{F}_s = q_s/m_s(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The EM fields are determined from Maxwell equations

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \\ \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mu_0 \mathbf{J}\end{aligned}$$

Question: Can we solve the VM system *efficiently* while conserving important invariants?

We know that the Vlasov-Maxwell system conserves, total number of particles; total (field + particle) momentum; total (field + particle) energy; other invariants. Can a numerical scheme be designed that retains (some or all) of these properties?

For understanding solar-wind turbulence and other problems, we would like a noise-free algorithm that allows studying phase-space cascades correctly, in a noise-free manner.

We use DG for both Vlasov and Maxwell equations

Start from Vlasov equation written as advection equation in phase-space:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{z}} \cdot (\boldsymbol{\alpha} f_s) = 0$$

where advection velocity is given by $\boldsymbol{\alpha} = (\mathbf{v}, q/m(\mathbf{E} + \mathbf{v} \times \mathbf{B}))$.

To derive the semi-discrete Vlasov equation using a discontinuous Galerkin algorithm, we introduce phase-space basis functions $w(\mathbf{z})$, and derive the discrete scheme:

$$\int_{K_j} w \frac{\partial f_h}{\partial t} \, d\mathbf{z} + \oint_{\partial K_j} w^- \mathbf{n} \cdot \hat{\mathbf{F}} \, dS - \int_{K_j} \nabla_{\mathbf{z}} w \cdot \boldsymbol{\alpha}_h f_h \, d\mathbf{z} = 0$$

We use DG for both Vlasov and Maxwell equations

Multiply Maxwell equations by basis φ and integrate over a cell. We have terms like

$$\int_{\Omega_j} \underbrace{\varphi \nabla \times \mathbf{E}}_{\nabla\times(\varphi\mathbf{E})-\nabla\varphi\times\mathbf{E}} \, d^3\mathbf{x}.$$

Gauss law can be used to convert one volume integral into a surface integral

$$\int_{\Omega_j} \nabla \times (\varphi \mathbf{E}) \, d^3\mathbf{x} = \oint_{\partial\Omega_j} d\mathbf{s} \times (\varphi \mathbf{E})$$

Using these expressions we can now write the discrete weak-form of Maxwell equations as

$$\int_{\Omega_j} \varphi \frac{\partial \mathbf{B}_h}{\partial t} \, d^3\mathbf{x} + \oint_{\partial\Omega_j} d\mathbf{s} \times (\varphi^- \hat{\mathbf{E}}_h) - \int_{\Omega_j} \nabla\varphi \times \mathbf{E}_h \, d^3\mathbf{x} = 0$$

$$\epsilon_0\mu_0 \int_{\Omega_j} \varphi \frac{\partial \mathbf{E}_h}{\partial t} \, d^3\mathbf{x} - \oint_{\partial\Omega_j} d\mathbf{s} \times (\varphi^- \hat{\mathbf{B}}_h) + \int_{\Omega_j} \nabla\varphi \times \mathbf{B}_h \, d^3\mathbf{x} = -\mu_0 \int_{\Omega_j} \varphi \mathbf{J}_h \, d^3\mathbf{x}.$$

Is energy conserved? Are there any constraints on basis functions/numerical fluxes?

Answer: Yes! If one is careful. We want to check if

$$\frac{d}{dt}\sum_j\sum_s\int_{K_j}\frac{1}{2}m|\mathbf{v}|^2 f_h\,d\mathbf{z}+\frac{d}{dt}\sum_j\int_{\Omega_j}\left(\frac{\epsilon_0}{2}|\mathbf{E}_h|^2+\frac{1}{2\mu_0}|\mathbf{B}_h|^2\right)\,d^3\mathbf{x}=0$$

Proposition

If central-fluxes are used for Maxwell equations, and if $|\mathbf{v}|^2$ is projected to the approximation space, the semi-discrete scheme conserves total (particles plus field) energy exactly.

The proof is rather complicated, and needs careful analysis of the discrete equations (See Juno et. al. JCP 2018)

Remark

If upwind fluxes are used for Maxwell equations, the total energy will decay monotonically. Note that the energy conservation does not depend on the fluxes used to evolve Vlasov equation.

Is momentum conserved?

Answer: No. Errors in momentum come about due to discontinuity in electric field at cell interfaces. However, *momentum conservation errors are independent of velocity space discretization, and drop rapidly with increasing configuration space resolution*.

Entropy increases monotonically

In order to correctly understand entropy production, one needs to ensure that discrete scheme either maintains or increase entropy in the collisionless case. We can show

Proposition

If the discrete distribution function f_h remains positive definite, then the discrete scheme grows the discrete entropy monotonically

$$\sum_j \frac{d}{dt} \int_{K_j} -f_h \ln(f_h) \geq 0$$

Some Parting Thoughts and All the Best!

Here are some personal parting thoughts on computational physics:

- Computational physics is a rapidly evolving field, with major investments being made across the world. Good field to be in!
- Strive for **technical excellence**. Do not settle for existing methods or tools and spend time in understanding deeply **both** the physics of the equations *and* the numerics used to solve them. Go beyond your classwork and thesis reasearch (make it a point to read arxiv physics.comp-ph and math.NA postings *every day*).
- Modern computational physics is moving to C++: please learn it. Use good software practices (write modular code, use version control, build systems, regression tests). Even for your thesis code!
- To become *really good* you must **apprentice yourself** to a genuine expert.
- If you become an expert at the (i) physics (ii) mathematics of the numerical methods (iii) programming and software techniques, you will be in a very strong position to contribute to development in many different fields. You will bring *unique skills* which few other people will be able to match.