Computational Methods in Plasma Physics. Lecture I

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Goal: Modern Computational Techniques for Plasma Physics

Vast majority of plasma physics in contained in the Vlasov-Maxwell equations that describe self-consistent evolution of distribution function $f(\mathbf{x}, \mathbf{v}, t)$ and electromagnetic fields:

$$\frac{\partial f_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (\mathbf{v} f_s) + \nabla_{\mathbf{v}} \cdot (\mathbf{F}_s f_s) = \left(\frac{\partial f_s}{\partial t} \right)_c$$

where $\mathbf{F}_s = q_s/m_s(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. The EM fields are determined from Maxwell equations

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} &= 0 \\ \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} &= -\mu_0 \sum_s q_s \int_{-\infty}^{\infty} v f_s \, d\mathbf{v}^3 \end{aligned}$$

Highly nonlinear: fields tell particles how to move. Particle motion generates fields. This is a very difficult system of equations to solve! Theoretical and computational plasma physics consists of making approximations and solving these equations in specific situations.

Why is solving Vlasov-Maxwell equations directly so hard?

Despite being the fundamental equation in plasma physics the VM equations remain highly challenging to solve.

- Highly nonlinear with the coupling between fields and particles via currents and Lorentz force. Collisions can further complicate things due to long-range forces in a plasma; dominated by small-angle collisions
- High dimensionality and multiple species with large mass ratios: 6D phase-space, $m_e/m_p = 1/1836$ and possibly dozens of species.
- Enormous scales in the system: light speed and electron plasma oscillations; cyclotron motion of electrons and ions; fluid-like evolution on intermediate scales; resistive slow evolution of near-equilibrium states; transport scale evolution in tokamak discharges. 14 orders of magnitude of physics in these equations!

Many approximations developed over the decades

Modern computational plasma physics consists of making justified approximations to the VM system and then coming up with efficient schemes to solve them.

- Major recent theoretical development in plasma physics is the discovery of gyrokinetic equations, an asymptotic approximation for plasmas in strong magnetic fields. Reduces dimensionality to 5D (from 6D) and eliminates cyclotron frequency and gyroradius from the system. Very active area of research.
- Many fluid approximations have been developed to treat plasma via low-order moments: extended MHD models; multimoment models; various reduced MHD equations
- Numerical methods for these equations have undergone renaissance in recent years: emphasis on *memetic* schemes that preserve conservation laws and some geometric features of the continuous equations. Based on Lagrangian and Hamiltonian formulation of basic equations. Very active area of research.

With advent of large scale computing much research is now focused on schemes that scale well on thousands (millions) of CPU/GPU cores.

Goal of this course is to look at some key schemes and study their properties

Computational plasma physics is vast: we can only cover a (very) small fraction of interesting methods. In this class we will focus on

- Solving the Vlasov-Maxwell equations using particles, the "Particle-in-cell" method; methods to solve Maxwell equations. This is probably the most widely used method that yields reasonable results for many kinetic problems
- Shock-capturing methods for plasma fluid equations. These are particularly relevant to astrophysical problems in which flows can be supersonic or super-Alfvenic. A brief look at fluid solvers for use in fusion machines (tokamaks, stellarators) in which dynamics is much slower.
- Directly discretizing the Vlasov-Maxwell equations as a PDE in 5D/6D. This is an emerging area of active research and may open up study of turbulence in fusion machines and also exploring fundamental plasma physics in phase-space.

Conservation properties of Vlasov-Maxwell equations

It is important to design methods that preserve at least some properties of continuous Vlasov-Maxwell system. Define the moment operator for any function $\varphi(\mathbf{v})$ as

$$\langle \varphi(\mathbf{v}) \rangle_s \equiv \int_{-\infty}^{\infty} \varphi(\mathbf{v}) f_s(t, \mathbf{x}, \mathbf{v}) \, d^3\mathbf{v}.$$

The Vlasov-Maxwell system conserves particles:

$$\frac{d}{dt} \int_\Omega \sum_s \langle 1 \rangle_s \, d^3\mathbf{x} = 0.$$

The Vlasov-Maxwell system conserves total (particles plus field) momentum.

$$\frac{d}{dt} \int_\Omega \left(\sum_s \langle m_s \mathbf{v} \rangle_s + \epsilon_0 \mathbf{E} \times \mathbf{B} \right) \, d^3\mathbf{x} = 0.$$

The Vlasov-Maxwell system conserves total (particles plus field) energy.

$$\frac{d}{dt} \int_\Omega \left(\sum_s \langle \frac{1}{2} m_s |\mathbf{v}|^2 \rangle_s + \frac{\epsilon_0}{2} |\mathbf{E}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) \, d^3\mathbf{x} = 0.$$

Conservation properties of Vlasov-Maxwell equations

Besides the fundamental conservation laws, in the *absence of collisions* we can also show that

$$\frac{d}{dt}\int_K \frac{1}{2} f_s^2 \, d\mathbf{z} = 0,$$

where the integration is taken over the complete phase-space. Also, the entropy is a *non-decreasing* function of time

$$\frac{d}{dt}\int_K -f_s \ln(f_s) \, d\mathbf{z} \geq 0.$$

For collisionless system the entropy remains *constant*. (Prove these properties as homework/classwork problems).

- It is not always possible to ensure all these properties are preserved numerically. For example: usually one can either ensure momentum *or* energy conservation but not both; it is very hard to ensure $f(t, \mathbf{x}, \mathbf{v}) > 0$.
- Much of modern computational plasma physics research is aimed towards constructing schemes that preserve these properties.