Find $\mathbf{A}$ *if* $\mathbf{A} = \mathbf{R} + \mathbf{A} \times \mathbf{B}$

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The basic problem and its solution. We want to find the vector $\mathbf{A}$ that satisfies the equation

$$\mathbf{A} = \mathbf{R} + \mathbf{A} \times \mathbf{B}. \tag{1}$$

Take the cross-product with $\mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \mathbf{R} \times \mathbf{B} + (\mathbf{A} \times \mathbf{B}) \times \mathbf{B} \tag{2}$$

Now rewrite the last term as

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \mathbf{A})\mathbf{B} - B^2\mathbf{A} \tag{3}$$

Follows from $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$. See the NRL Plasma Formulary, for example.

to get

$$\mathbf{A} \times \mathbf{B} = \mathbf{R} \times \mathbf{B} + (\mathbf{B} \cdot \mathbf{A})\mathbf{B} - B^2\mathbf{A}. \tag{4}$$

From the original equation we have $\mathbf{A} \times \mathbf{B} = \mathbf{A} - \mathbf{R}$ and $\mathbf{B} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{R}$. Plugging these into the last expression we finally get

$$\mathbf{A} = \frac{\mathbf{R} + \mathbf{R} \times \mathbf{B} + (\mathbf{B} \cdot \mathbf{R})\mathbf{B}}{1 + B^2}. \tag{5}$$

We can write this in an alternate way as follows. Let $\mathbf{b} = \mathbf{B}/B$ be the unit vector along $\mathbf{B}$. Then, every vector can be decomposed into parallel and perpendicular components, for example $\mathbf{A} = A_\parallel \mathbf{b} + \mathbf{A}_\perp$, where $A_\parallel = \mathbf{A} \cdot \mathbf{b}$. From Eq. (1), we see that $A_\parallel = R_\parallel$. Decomposing the right-hand of Eq. (5) into parallel and perpendicular components, we get

$$\mathbf{A}_\perp = \frac{\mathbf{R}_\perp + \mathbf{R}_\perp \times \mathbf{B}}{1 + B^2}, \tag{6}$$

which is the solution to the equation $\mathbf{A}_\perp = \mathbf{R}_\perp + \mathbf{A}_\perp \times \mathbf{B}$. This form is often more useful than Eq. (5).

Particle in a static electromagnetic field. We want to discretize the ODE describing the motion of a charged particle in a given EM field.

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{7}$$

We will use a time-centered implicit scheme

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\Delta t} = \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}^{n+1} + \mathbf{v}^n}{2} \times \mathbf{B}\right). \tag{8}$$

Defining $\mathbf{v}^{n+1/2} \equiv (\mathbf{v}^{n+1} + \mathbf{v}^n)/2$ we can write this as

$$\mathbf{v}^{n+1/2} = \mathbf{v}^n + \frac{q\Delta t}{2m}\mathbf{E} + \mathbf{v}^{n+1/2} \times \frac{q\Delta t\mathbf{B}}{2m}. \tag{9}$$

This is in the same form as Eq. (1), and using Eq. (5) we can write the explicit update formula

$$\mathbf{v}^{n+1/2} = \lambda\left(\mathbf{v}^n + \frac{q\Delta t}{2m}\mathbf{v}^n\times\mathbf{B} + \frac{q^2\Delta t^2}{4m^2}(\mathbf{v}^n\cdot\mathbf{B})\mathbf{B}\right) + \frac{\lambda q\Delta t}{2m}\left(\mathbf{E} + \frac{q\Delta t}{2m}\mathbf{E}\times\mathbf{B} + \frac{q^2\Delta t^2}{4m^2}(\mathbf{E}\cdot\mathbf{B})\mathbf{B}\right) \tag{10}$$

where $\lambda = 1/(1+q^2\Delta t^2 B^2/4m^2)$. Once $\mathbf{v}^{n+1/2}$ is determined, we can calculate $\mathbf{v}^{n+1} = 2\mathbf{v}^{n+1/2} - \mathbf{v}^n$.

Source terms for five-moment equations. A more complex example are the source terms for the five-moment equations

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$$\frac{d\mathbf{v}_s}{dt} = \frac{q_s}{m_s}\left(\mathbf{E} + \mathbf{v}_s\times\mathbf{B}\right) \tag{11}$$

$$\epsilon_0\frac{d\mathbf{E}}{dt} = -\sum_s q_s n_s \mathbf{v}_s \tag{12}$$

where, for the plasma species s, n_s is the number density, $\mathbf{v}_s$ is the velocity, q_s and m_s are the charge and mass respectively. Further, $\mathbf{E}$ is the electric field, and ϵ_0 is permittivity of free space. In these equations the magnetic field and number density are constants, as there are no source terms for these quantities.

It is more convenient to work in terms of the plasma current $\mathbf{J}_s \equiv q_s n_s \mathbf{v}_s$, and normalized electric field $\mathbf{F} = \epsilon_0\mathbf{E}$ which leads to the coupled system

$$\frac{d\mathbf{J}_s}{dt} = \omega_s^2\mathbf{F} + \mathbf{J}_s\times\Omega_s\mathbf{b} \tag{13}$$

$$\frac{d\mathbf{F}}{dt} = -\sum_s \mathbf{J}_s \tag{14}$$

where $\Omega_s \equiv q_s B/m_s$ is the species cyclotron frequency, $\mathbf{b} = \mathbf{B}/B$ is the unit vector along the magnetic field and $\omega_s \equiv \sqrt{q_s^2 n_s/m_s}$ is the species plasma frequency. This is a system of linear, constant-coefficient ODEs for the $3s+3$ unknowns $\mathbf{J}_s$ and $\mathbf{F}$.

To solve the system of ODEs we replace the time-derivatives with time-centered differences. This leads to the discrete equation

In `Gkeyll` this system of linear equations is solved using a linear algebra package. The matrix which is inverted is $(3s+3)\times(3s+3)$ in size. The explicit solution below may be faster than inverting a matrix.

$$\frac{\mathbf{J}_s^{n+1/2} - \mathbf{J}_s^n}{\Delta t/2} = \omega_s^2\mathbf{F}^{n+1/2} + \mathbf{J}_s^{n+1/2}\times\Omega_s\mathbf{b} \tag{15}$$

$$\frac{\mathbf{F}^{n+1/2} - \mathbf{F}^n}{\Delta t/2} = -\sum_s \mathbf{J}_s^{n+1/2} \tag{16}$$

where $\mathbf{J}_s^{n+1/2} = (\mathbf{J}_s^{n+1} + \mathbf{J}_s^n)/2$ and $\mathbf{F}^{n+1/2} = (\mathbf{F}^{n+1} + \mathbf{F}^n)/2$.

We will solve the parallel and perpendicular components of these equations separately. Consider the parallel part first

$$J_{s\|}^{n+1/2} = J_{s\|}^n + \frac{\Delta t}{2}\omega_s^2 F_\|^{n+1/2} \tag{17}$$

$$F_\|^{n+1/2} = F_\|^n - \frac{\Delta t}{2}\sum_s J_{s\|}^{n+1/2}. \tag{18}$$

There are two coupled linear scalar equations and can be solved easily. Eliminating $J_{s\|}$ in the second equation we get

$$F_{\|}^{n+1/2} = \gamma F_{\|}^{n} - \frac{\gamma \Delta t}{2} \sum_s J_{s\|}^{n} \tag{19}$$

where $\gamma = (1 + \Delta t^2 \sum_s \omega_s^2/4)^{-1}$. Using this expression in the first equation allows us to calculate $J_{s\|}^{n+1/2}$. This completes the solution of the parallel system.

The perpendicular system is

$$\mathbf{J}_{s\perp}^{n+1/2} = \mathbf{J}_{s\perp}^{n} + \frac{\Delta t}{2}\omega_s^2 \mathbf{F}_{\perp}^{n+1/2} + \mathbf{J}_{s\perp}^{n+1/2} \times \frac{\Delta t \Omega_s \mathbf{b}}{2} \tag{20}$$

$$\mathbf{F}_{\perp}^{n+1/2} = \mathbf{F}_{\perp}^{n} - \frac{\Delta t}{2} \sum_s \mathbf{J}_{s\perp}^{n+1/2}. \tag{21}$$

The first of this equation is in our "standard" form and can be solved as

$$\mathbf{J}_{s\perp}^{n+1/2} = \mathbf{J}_{s\perp}^{*} + \frac{\lambda_s \omega_s^2 \Delta t}{2} \left(\mathbf{F}_{\perp}^{n+1/2} + \mathbf{F}_{\perp}^{n+1/2} \times \frac{\Delta t \Omega_s \mathbf{b}}{2} \right) \tag{22}$$

where $\lambda_s = (1 + \Delta t^2 \Omega_s^2/4)$ and

$$\mathbf{J}_{s\perp}^{*} \equiv \lambda_s \left(\mathbf{J}_{s\perp}^{n} + \mathbf{J}_{s\perp}^{n} \times \frac{\Delta t \Omega_s \mathbf{b}}{2} \right). \tag{23}$$

Substituting Eq. (22) in Eq. (21) we get

$$\mathbf{F}_{\perp}^{n+1/2} = \mathbf{R}_{\perp} + \mathbf{F}_{\perp}^{n+1/2} \times \alpha \mathbf{b} \tag{24}$$

where

$$\mathbf{R}_{\perp} \equiv \frac{\mathbf{F}_{\perp}^{n} - \Delta t/2 \sum_s \mathbf{J}_{s\perp}^{*}}{1 + \Delta t^2/4 \sum_s \lambda_s \omega_s^2} \tag{25}$$

and

$$\alpha \equiv \frac{\Delta t^3/8 \sum_s \lambda_s \omega_s^2 \Omega_s}{1 + \Delta t^2/4 \sum_s \lambda_s \omega_s^2}. \tag{26}$$

Again, this is in the standard form, and we can write

$$\mathbf{F}_{\perp}^{n+1/2} = \frac{\mathbf{R}_{\perp} + \mathbf{R}_{\perp} \times \alpha \mathbf{b}}{1 + \alpha^2}. \tag{27}$$

This determines $\mathbf{F}_{\perp}^{n+1/2}$ in terms of the previous time-step values. Using in Eq. (22) allows us to compute the currents, completing the solution.