PROBLEMS

4-2a Sketch the locations of a, v, and x for the alternative integrator,

$$\frac{v_{t+\Delta t} - v_t}{\Delta t} = \frac{1}{2}(3a_t - a_{t-\Delta t}) \tag{11}$$

$$\frac{x_{t+\Delta t} - x_t}{\Delta t} = \frac{1}{2}(v_{t+\Delta t} + v_t) \tag{12}$$

Show that this integrator produces a frequency for the simple harmonic oscillator (1) of

$$\omega \approx \omega_0\left[1 + \frac{1}{6}(\omega_0\Delta t)^2 + i\frac{(\omega_0\Delta t)^3}{8} + \cdots\right] \tag{13}$$

Hint: Use $x_{t\pm n\Delta t} = z^{\pm n}x_t$, $z \equiv \exp(-i\omega\Delta t)$ and write (11) and (12) as a matrix. Set the determinant of the coefficients equal to zero to produce an equation in z(cubic) to be solved. The phase error is four times larger than that of the leap-frog scheme, and there is mild growth. Show how (with a sketch, as in Figure 2-4a) Δt can be halved or doubled in one step. This method has the disadvantage of requiring storage of the previous acceleration $a_{t-\Delta t}$ in addition to previous velocity v_t and position x_t.

4-2b Discuss the possible use of *integer arithmetic* (no floating point) in the mover, with some care as to the number of bits needed for reasonable accuracy. Keep in mind that small changes are lost; that is, nothing happens unless $v\Delta t$ exceeds the least step in x and $a\Delta t$ exceeds the least velocity. *Hint*: 17 bits is marginal or fatal in quiet start. This coding was exploited by *Estabrook and Tull* (1980) for very high speed, almost twice as fast on the CDC-7600 (in machine language) as ES1 is on the CRAY-I (in FORTRAN)!

4-3 NEWTON-LORENTZ FORCE; THREE-DIMENSIONAL $\mathbf{v \times B}$ INTEGRATOR

The particle equations of motion to be integrated are

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{1}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{2}$$

We desire a centered-difference form of the Newton-Lorentz equations of motion. The magnetic term is centered by averaging $\mathbf{v}_{t-\Delta t/2}$ and $\mathbf{v}_{t+\Delta t/2}$, following *Buneman* (1967). The other terms are treated as before. Hence, (1) becomes

$$\frac{\mathbf{v}_{t+\Delta t/2} - \mathbf{v}_{t-\Delta t/2}}{\Delta t} = \frac{q}{m}\left[\mathbf{E} + \frac{\mathbf{v}_{t+\Delta t/2} + \mathbf{v}_{t-\Delta t/2}}{2}\times\mathbf{B}\right] \tag{3}$$

This vector equation for $\mathbf{v}_{t+\Delta t/2}$ can be solved as three simultaneous scalar equations, one for each component. Instead, we choose to obtain a simpler solution using several steps.

The first method (*Buneman,* 1967) is to subtract the drift velocity $\mathbf{E}\times\mathbf{B}/B^2$ from $\mathbf{v}$, as

$$\mathbf{v}'_{\text{old}} = \mathbf{v}_{t-\Delta t/2} - \frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

$$\mathbf{v}'_{\text{new}} = \mathbf{v}_{t+\Delta t/2} - \frac{\mathbf{E}\times\mathbf{B}}{B^2}$$

Similar to (1), this leaves just a rotation of $\mathbf{v}_\perp$ and free acceleration

$$\frac{\mathbf{v}'_{\text{new}} - \mathbf{v}'_{\text{old}}}{\Delta t} = \frac{q}{m}\left[\mathbf{E}_\parallel + \frac{\mathbf{v}'_{\text{new}} + \mathbf{v}'_{\text{old}}}{2}\times\mathbf{B}\right]$$

We discuss the rotation in Problem 4-3a and Section 4-4.

Another method separates the electric and magnetic forces c (*Boris*, 1970b). Substitute

$$\mathbf{v}_{t-\Delta t/2} = \mathbf{v}^- - \frac{q\mathbf{E}}{m}\frac{\Delta t}{2}$$

$$\mathbf{v}_{t+\Delta t/2} = \mathbf{v}^+ + \frac{q\mathbf{E}}{m}\frac{\Delta t}{2}$$

into (3); then $\mathbf{E}$ cancels entirely (not just $\mathbf{E}_\perp$), which leaves

$$\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} = \frac{q}{2m}(\mathbf{v}^+ + \mathbf{v}^-)\times\mathbf{B}$$

which is a rotation (see Problem 4-3a). The steps to compute are the electric impulse to $\mathbf{v}_{t-\Delta t/2}$ using (7) to obtain $\mathbf{v}^-$; rotate accord to obtain $\mathbf{v}^+$, and add the remaining half of the electric impulse (8) $\mathbf{v}_{t+\Delta t/2}$. These are the same steps, motivated differently, as in Se Separation of parallel and perpendicular components is not nee Boris' method, and the relativistic generalization is straightforward.

Finally, we check the angle of rotation θ which we expect to b $\omega_c\Delta t = qB\Delta t/m$. By inspection of Figure 4-3a, we see that

$$\left|\tan\frac{\theta}{2}\right| = \frac{|\mathbf{v}_\perp^+ - \mathbf{v}_\perp^-|}{|\mathbf{v}_\perp^+ + \mathbf{v}_\perp^-|} = \frac{qB}{m}\frac{\Delta t}{2} = \frac{\omega_c\Delta t}{2}$$

where we have used (9) in the last step. Hence, the difference equ produces a rotation through angle

$$\theta = 2\arctan\left(\frac{qB}{m}\frac{\Delta t}{2}\right) = \omega_c\Delta t\left(1 - \frac{(\omega_c\Delta t)^2}{12} + \cdots\right)$$

which has less than one percent error for $\omega_c\Delta t < 0.35$.

PROBLEMS

4-3a Show that (9) is only a *rotation* of $\mathbf{v}$. *Hint*: take the scalar product of (9) with (

4-3b Consider a model with $\mathbf{B}_0$ uniform and $\mathbf{E}_\perp = 0$, in which a particle at speed v motion in the $\mathbf{x}_\perp$ plane. Let the orbit be followed by (subscripts refer to time steps)

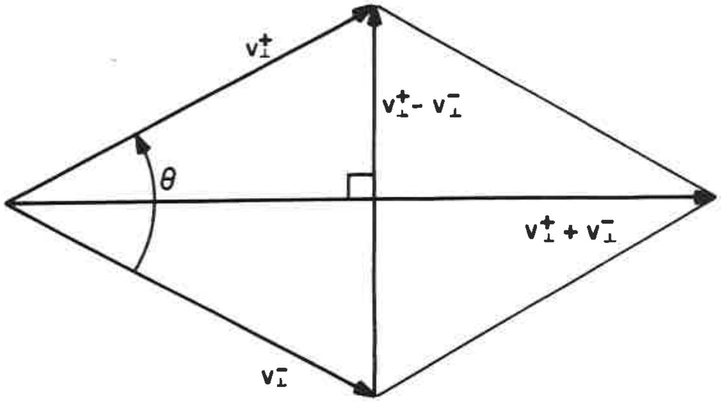

Figure 4-3a Knowing that (9) represents a rotation, we construct this diagram, from which $\tan(\theta/2)$ is readily obtained.

$$\frac{\mathbf{x}^+ - \mathbf{x}^0}{\Delta t} = \mathbf{v}^+ \qquad \frac{\mathbf{x}^0 - \mathbf{x}^-}{\Delta t} = \mathbf{v}^-$$

with $\mathbf{v}^+$ obtained from

$$\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} = \lambda\left(\frac{\mathbf{v}^+ + \mathbf{v}^-}{2}\right) \times \left(\frac{q\mathbf{B}}{m}\right)$$

as shown in Figure 4-3b. Using $\tan\alpha$ from this figure [*Hint*: see (10)], show that

$$\lambda = \frac{\tan\alpha}{\frac{1}{2}\omega_c \Delta t}$$

which is $(\tan\alpha)/\alpha$ if we ask that the mover produce the correct gyrophase, $\alpha = \omega_c \Delta t/2$. Next, requiring that the gyroradius and period be reproduced correctly, show that

$$|\mathbf{v}^+| = |\mathbf{v}^-| = v\left(\frac{\sin\alpha}{\alpha}\right)$$

Last, show that when $\mathbf{E}_\perp$ is included, the λ multiplier appears as $\lambda(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B})$ in order to produce the correct $\mathbf{E} \times \mathbf{B}/B^2$ drift. [These ideas came from *Hockney* (1966), *Buneman* (1967), and R. H. Gordon (unpublished Berkeley seminar, 1971).]

4-3c (Due to R. H. Gordon). Using any of the $\mathbf{v} \times \mathbf{B}$ integrators given here, for uniform $B_0\hat{z}$, plot particle orbits in the x, y plane:
(a) Without the $\tan\alpha/\alpha$ correction, supposedly with a integral number of steps per cyclotron period; note where points on the second and succeeding cycles are with respect to those on the first.
(b) Repeat (a) with $\tan\alpha/\alpha$ correction; try $0 < \alpha < \pi$.
(c) Like (b), with $\omega_c \Delta t = 2\pi$; compare with true orbit; explain the motion; is this instability?
(d) Like (b), with $\omega_c \Delta t = 2\pi - \delta$; compare with true orbit; explain the motion.

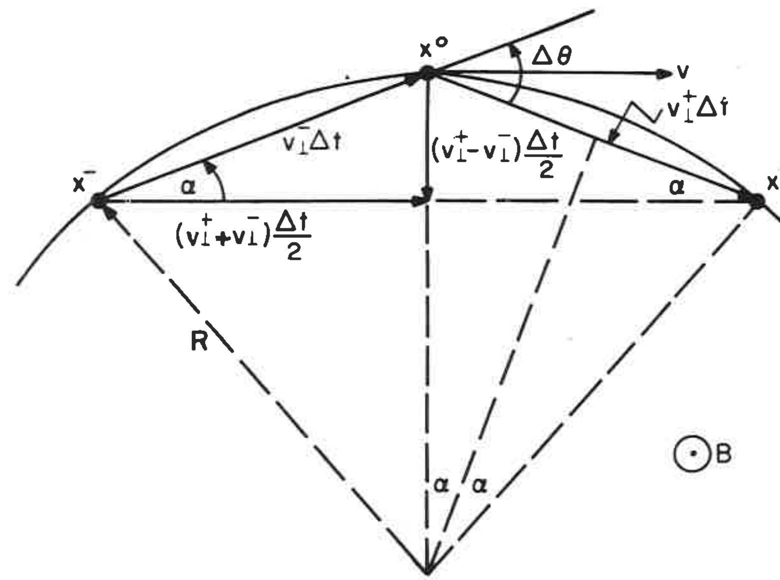

Figure 4-3b Velocities and positions in the plane normal to the uniform magnetic f
$\mathbf{E}_\perp = 0$ in which the particle orbit is a circle (cyclotron motion). The compu
difference orbit is made up of straight line segments connecting old and new positio

4-4 IMPLEMENTATION OF THE v × B ROTATION

First consider the case in which **B** is parallel to the z axis. I
plane the rotation is through an angle θ where

$$\tan\frac{\theta}{2} = -\frac{qB}{m}\frac{\Delta t}{2}$$

This gives a good approximation to rotation angle θ when θ is not
[4-3(11)], and is convenient when B is not constant. In ES1, B i
we evaluate $\tan\theta/2 = -\tan(qB\Delta t/2m)$; obtaining the correc
angle costs nothing more.

Now we use this value of $\tan\theta/2$ in the half-angle formulas
$\cos\theta$ and $\sin\theta$ for the velocity rotation. Letting

$$t = -\tan\frac{\theta}{2}$$

we have

$$s \equiv -\sin\theta = \frac{2t}{1 + t^2}$$

$$c \equiv \cos\theta = \frac{1 - t^2}{1 + t^2} \tag{4}$$

The rotation becomes

$$v_x^+ = cv_x^- + sv_y^- \tag{5}$$

$$v_y^+ = -sv_x^- + cv_y^- \tag{6}$$

The mover requires no evaluation of transcendental functions, which is a significant time saving when B is not constant. Equations (3) to (6) require 7 multiplies, 1 divide, and 5 adds. *Buneman* (1973) reduces this to:

$$v'_x = v_x^- + v_y^- t \tag{7}$$

$$v_y^+ = v_y^- - v'_x s \tag{8}$$

$$v_x^+ = v'_x + v_y^+ t \tag{9}$$

with 4 multiplies, 1 divide, and 5 adds. The saving of 3 multiplies per particle per time step is desirable.

When the directions of **B** and **v** are arbitrary, a convenient rotation in vector form is described by *Boris* (1970). First $\mathbf{v}^-$ is incremented to produce a vector $\mathbf{v}'$ which is perpendicular to $(\mathbf{v}^+ - \mathbf{v}^-)$ and **B** (see Figure 4-4a).

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t} \tag{10}$$

The angle between $\mathbf{v}^-$ and $\mathbf{v}'$ is just $\theta/2$, therefore the vector **t** is seen from Figure 4-4a to be given by

$$\mathbf{t} \equiv -\hat{\mathbf{b}} \tan\frac{\theta}{2} = \frac{q\mathbf{B}}{m}\frac{\Delta t}{2} \tag{11}$$

Finally, $\mathbf{v}^+ - \mathbf{v}^-$ is parallel to $\mathbf{v}' \times \mathbf{B}$, so

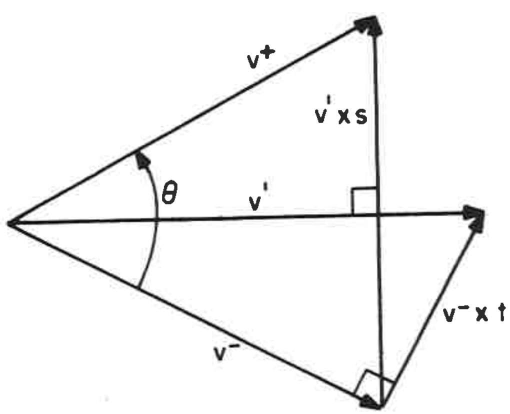

Figure 4-4a Velocity space showing the rotation from $\mathbf{v}^-$ to $\mathbf{v}^+$. The velocities shown are projections of the total velocities onto the plane perpendicular to **B**.

$$\mathbf{v}^+ = \mathbf{v}^- + \mathbf{v}' \times \mathbf{s}$$

where s is parallel to **B** and its magnitude is determined by the r
$|\mathbf{v}^-|^2 = |\mathbf{v}^+|^2$

$$\mathbf{s} = \frac{2\mathbf{t}}{1 + t^2}$$

Boris' algorithm is readily made relativistic; see Chapter 15.

PROBLEMS

4-4a Verify that (7) - (9) has the same result as (5) and (6).

4-4b Using $|\mathbf{v}^+| = |\mathbf{v}^-|$ (a rotation), obtain s (13).

4-4c Show that Boris' rotation satisfies the equation of motion 4-3(9), if $\mathbf{t} = q\mathbf{B}\Delta t$

4-5 APPLICATION TO ONE-DIMENSIONAL PROGRA

In programs with one dimension x and with *two velocities*, v_x a allow a magnetic field B_z, the motion is all perpendicular to **B**. He the vector equations of the previous sections, we obtain the 1 rotate-accel algorithm in Section 2-4.

A program with one dimension and with *three velocities* 1d3v up as shown in Figure 4-5a. Let $\mathbf{B}_0$ (constant, uniform) be in the and make an angle θ with the z axis. The self-consistent **E** and along x, normal to the sheets. If we stopped here, then the pe motion in z could be ignored ($F_z = 0$). However, on occasion interested in applying an electric field E_{ext} along y which produces in turn produces an $F_z = -qv_yB_x$ and a z drift, $(v_E)_z = -(E_{\text{ext}})_y$ point of the exercise is to include both $v_\parallel$ and $v_\perp$ as well as $k_\parallel$ and model. It is convenient to have the motion solved for in the p perpendicular directions, which leads to the invention of the x' co $\perp$ to $\mathbf{B}_0$, at angle θ with respect to x). Call the field due to th $\mathbf{E}_{\text{self-consistent}} \equiv \mathbf{E}_{\text{sc}}$. Then the fields are:

$$\mathbf{E}_{\text{sc}} = \hat{\mathbf{x}}E_{\text{sc}} = \hat{\mathbf{x}}'E_{\text{sc}}\cos\theta + \hat{\mathbf{b}}_0 E_{\text{sc}}\sin\theta$$

$$\mathbf{E}_{\text{ext}} = \hat{\mathbf{y}}E_{\text{ext}}$$

$$\mathbf{B} = \mathbf{B}_0$$

The equations of motion in the (x', y, B_0) coordinates are integra accel-rot-accel method, as follows

$$v_{x_1'} = v_{x'}(t - \Delta t/2) + \frac{q}{m}\frac{\Delta t}{2}E_{\text{sc}}\cos\theta$$